DYNAMICS OF PLASTIC DEFORMATIONS IN A BAR EXHIBITING STRAIN-RATE EFFECT AND SUBJECTED TO ALTERNATING STRESSES

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ABSTRACT. Dynamics of the vibration of a bar exhibiting strain-rate effect when subjected to an alternating mechanical stress has been worked out, following the operational method. Expression for the amplitude of vibration at the free end in terms of resonance frequency and fractional detuning at 3 db point has been derived. An accurate formula is also derived for the resonance frequency. An expression giving the displacement at any point along the bar has also been derived. The work of previous authors have also been critically discussed.

INTRODUCTION

The theory of the extensional vibrations of a bar excited by impact of a 'rigid' load has been worked out by a number of workers. (Boussinesq, 1885 ; Ghosh, 1936 ; Ghosh and Dhar, 1930). Ghosh and Ghosh (1950) extended the case, applying the powerful operational method, for an elastic load struck at the free end of a bar, the other end being fixed. In a subsequent paper (Ghosh and Ghosh, 1952) dealing with the transmission of stress under the conditions of rapidly applied loading, the author has calculated the time of collision during impact for each epoch and has derived many important conclusions therefrom. But in formulating the earlier theories the effect of internal friction in solids was neglected. Thereupon the author has built up the complete dynamics of the elastic strain-waves due to axial impulse and has extended the theory to cover plastic behaviour exhibited by most metals. In formulating all these theories the author has always used the powerful operational method, which is also employed here for solving the present problem. In the present theory the effect of viscosity, which arises out of internal friction, in metals on the transmission of strain-waves through a solid bar subjected to an alternating mechanical stress has been considered. The characteristic of viscosity is that in a moving fluid, the stresses differ from a state of pressure uniform in all directions about a point, by quantities depending on the 'rates of strain'.

It is usually assumed that these quantities are linear functions of the rates of strain. At least in the high polymers, a complete model of the solid must comprise an indefinitely large number of elements all obeying this linear model, but with differing values of appropriate elastic and viscous moduli. From our present standpoint this is sufficiently justified by the fact that the strain-velocities are regarded as infinitely small.

THEORY

For mathematical convenience, it is common in problems on vibrations to consider the relation between the stress, p, and the strain, e, in a solid to be of the form :

$$p = Ee + \eta \frac{de}{dt}$$

where E and η are the associated elastic and viscous moduli.

The equation of longitudinal motion in a thin bar or tube (Cady, 1922) exhibiting strain-rate effect, is then :

$$\rho \frac{d^2\omega}{dt^2} = E\frac{d^2\omega}{ds^2} + \eta\frac{d^3\omega}{dt.ds^2} \quad \ldots \quad (1)$$

where s = longitudinal co-ordinate,
ω = longitudinal displacement,
ρ = density of the bar,
E = Young's modulus of the bar,
and η = associated viscosity coefficient of the bar.

Equation (1) is equivalent to :

$$\frac{d^2\omega}{ds^2} = \frac{D_1{}^2\omega}{C^2} \quad \ldots \quad (1.1)$$

where
$$D_1 = D[1 + \eta_1 D]^{-1/2}$$
$$C^2 = E/\rho$$
$$\eta_1 = \eta/E \text{ and } D = \text{the operator } \frac{d}{dt}.$$

The general solution of equation (1.1) is

$$\omega = A \cosh \frac{D_1 s}{C} + B \sinh \frac{D_1 s}{C} \quad \ldots \quad (1.2)$$

If a periodic force of any period be applied at one end of the bar, any portion of it will vibrate with the same periodicity as the force, so that in the steady state,

$$\omega_s = \psi(s) . \frac{D}{D - in} \quad \ldots \quad (1.21)$$

expressed in operational form in which n represents the frequency of vibration.

At the free end of the bar, there being no external force, we have,

$$\text{at } s = 0, \ \frac{d\omega_s}{ds} = 0 \quad \ldots \quad (1.3)$$

Also, the displacement at the free end ($s=0$) may be expressed as

$$u_0 = \psi(s)_{s=0} \cdot \frac{D}{D-in} \quad \ldots \quad (1.4)$$

With the help of equations (1.3) and (1.4), equation (1.2) becomes :

$$\omega_s = \psi(s)_{s=0} \cosh \frac{D_1 s}{c} \cdot \frac{D}{D-in} \quad \ldots \quad (2)$$

The operational solution of equation (2) yields :

$$\omega_s = \psi(s)_{s=0}\left[\cosh^2\frac{\beta s}{c} - \sin^2\frac{ns}{c}(4-3\alpha)\right]^{1/2} e^{i(nt+\epsilon)} \quad \ldots \quad (2.1)$$

where

$$\alpha = 1 + \tfrac{1}{8}\eta_1^2 n^2, \quad \beta = \tfrac{1}{2}\eta_1 n^2 \quad \ldots \quad (2.11)$$

and

$$\tan\epsilon = \tanh\frac{\beta s}{c} \cdot \tan\frac{ns}{c}(4-3\alpha) \quad \ldots \quad (2.12)$$

If the bar of length l be excited at one end ($s=l$) by a periodic stress $F.e^{i(nt-\epsilon_1)}$ of peak value F, the equation of motion at $s=l$ is

$$E\left(\frac{d\omega}{ds}\right)_{s=l} + \eta D\left(\frac{d\omega}{ds}\right)_{s=l} = F.e^{i(nt-\epsilon_1)} \quad \ldots \quad (3)$$

Substituting values of $\left(\frac{d\omega}{ds}\right)_{s=l}$ from equation (2), equation (3) becomes,

$$F.e^{i(nt-\epsilon_1)} = \frac{E}{c}\psi(s)_{s=0} D[1+\eta_1 D]^{1/2}\sinh\frac{D_1 l}{c} . e^{int}$$

$$\frac{E}{c}\psi(s)_{s=0}[\alpha^2 n^2+\beta^2]^{1/2}\left\{\cosh^2\frac{\beta l}{c} - \cosh^2\frac{nl}{c}(1-3\alpha)\right\}^{1/2} e^{i(nt+\epsilon_1)} \quad \ldots \quad (3.1)$$

where $\epsilon_1 = \epsilon + \phi + \pi$, and $\tan\phi = \dfrac{\tanh\frac{\beta l}{c}\cot\frac{nl}{c}(4-3\alpha) - \frac{\beta}{\alpha n}}{\frac{\beta}{\alpha n}\tanh\frac{\beta l}{c}\cot\frac{nl}{c}(4-3\alpha)+1}$

at $s=l$.

The amplitude of vibration at the free end $s=0$ is, therefore, given by

$$\psi(s)_{s=0} = \frac{F}{\frac{E}{c}[\alpha^2 n^2+\beta^2]^{1/2}\left[\cosh^2\frac{\beta l}{c} - \cos^2\frac{nl}{c}(4-3\alpha)\right]^{1/2}} \quad \ldots \quad (4)$$

Equation (4) is in agreement with the result given by Lethersich and Pelzer (1950) upto terms containing $\eta_1^2 n^2$.

Also from equation (2.1) we get the displacement at any point along the bar in terms of the amplitude of vibration at the free end and is given by

$$|\omega_s| = \psi(s)_{s=0}\left[\cosh^2\frac{\beta s}{c} - \sin^2\frac{ns}{c}(4-3\alpha)\right]^{1/2} \quad \ldots \quad (5)$$

Differentiating the expression under the radical sign in equation (4) with respect to n and equating to zero, the maxima of the amplitude are found to occur when,

$$n=\frac{S\pi c}{l}\left(1+\frac{3}{8}\eta_1{}^2n^2\right)^{-1} \qquad \ldots \quad (6)$$

where $S=1,2,3 \ldots$ etc.

Similary, an inspection of equation (5) shows that minimum values of displacement (nodes) occur at distances given by

$$\frac{ns}{c}\left(1-\frac{3}{8}\eta_1{}^2n^2\right)=S'\frac{\pi}{2} \qquad \ldots \quad (7)$$

where $S'=1,3,5 \ldots$ etc.

DISCUSSIONS

Experiments (Lethersich, *et al*, 1950) made on a number of plastics indicate that at a frequency of 2×10^3c/s, $\eta_1{}^2n^2$ lies between 0.00016 and 0.003. Thus if $\eta_1{}^2n^2<0.00016$, the minimum value for most plastics, $(4-3\alpha)$ of equations (4) and (5) approaches unity.

Under this condition equation (6) shows that resonances occur at or near multiples of frequency when $n=n_r$,

where $$n_r=\frac{S\pi c}{l},\ S=1,2,3 \ldots \text{etc.}$$

Therefore expanding the cosh and cosine terms in equation (4) and using the approximate relations :

$$\cosh\theta=1+\frac{\theta^2}{2}+\frac{\theta^4}{24},$$

$$\cos\theta=1-\frac{\theta^2}{2}+\frac{\theta^4}{24},$$

a simplified expression for the amplitude near the resonance peaks is given by

$$\psi(s)_{s=0}=\frac{2F}{S\pi(\rho E)^{1/2}}\cdot\left[\frac{n^6\eta_1{}^2}{2n_r{}^2}+\frac{4n^2(n-n_r)^2}{n_r{}^2}\cdot\left(1-\frac{\eta_1{}^2n^2}{8}\right)\right.$$
$$\left.-\frac{4S^2\pi^2n^2(n-n_r)^4}{3n_r{}^4}\left(1-\frac{\eta_1{}^2n^2}{2}\right)\right]^{1/2} \qquad \ldots \quad (4.1)$$

Equation (4.1) gives the resonance amplitude at the free end, in terms of the harmonic number S and the fractional detuning. The resonant amplitude can, however, be found more accurately by using the relation (6) *i.e.*, replacing n in equation (4.1) by

$$\frac{S\pi c}{l}\left(1+\frac{3}{8}\eta_1{}^2n^2\right)^{-1}.$$

Relation (6), therefore, gives an accurate value for the resonance frequency and indicates that some sources of error creep in due to successive approximation in deriving a similar relation by Lethersich and Pelzer.

Relation (6) thus amends Lethersich and Pelzer's similar relation and improves the accuracy of Wegel and Walther's (1935) expression giving the band-width at the 3 db. point.

Equation (7) gives minimum values of displacement at distances given by

$$\frac{2ns}{c} = S'\pi, \quad S' = 1, 3, 5 \ldots \text{etc.}$$

when $\eta_1^2 n^2 < 0.00016$. This is also in agreement with Lethersich and Pelzer's relation giving the position of nodes.

With the above approximation, the displacement at the centre of the bar vibrating in its simplest mode at resonance is

$$\omega_{l/2} = \psi(s)_{s=0} \sinh \frac{\eta_1 n^2 l}{4c} \qquad \ldots \quad (5.1)$$

which is the only minimum value at the fundamental. The displacement at the centre of the bar for the same mode, when the amplitude of vibration at the free end has dropped by a factor of $\sqrt{2}$ (*i.e.*, 3 db.) on either side of the peak is obtained by putting $(n_r + \Delta)$ for n in equation (5), Δ being the band-width at the 3 db. point. This gives

$$\omega = \psi(s)_{s=0} \left[\cosh^2 \frac{\eta_1 n_r^2 l}{4c} \left(1 + \frac{\Delta}{n_r}\right)^2 - \cosh^2 \frac{\Delta l}{2c} \right]^{1/2} \qquad (5.2)$$

Neglecting $\frac{\Delta}{n_r}$ compared to unity and using Wegel and Walther's relation,

$\Delta = \frac{\eta_1 n_x^2}{2}$ at 3 db. point, we get,

$$\omega_\cdot \simeq \sqrt{2}\, \omega_{s=\frac{l}{2}} \qquad \ldots \quad (5.3)$$

Equation (5.3) is also in agreement with Lethersich and Pelzer's similar expression.

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